ON THE LAW OF DIURNAL ROTATION OF THE OPTICAL FIELD OF THE SIDEROSTAT AND HELIOSTAT.

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THE well-known instruments known as the *Heliostat* and the *Siderostat* permit a beam of light coming from a heavenly body sharing the diurnal motion to be sent in a constant direction by means of a movable mirror. The geometrical theory of these instruments is very simple; the heavenly body is considered as a luminous point and the incident beam is regarded as a straight line which in twenty-four hours describes a cone of revolution around the polar axis of the instrument, which is itself parallel to the Earth's axis. In order to secure the fixity of the reflected beam it is necessary and sufficient that the normal to the mirror shall remain parallel to the bisector of the angle between the ray coming from the heavenly body and a given fixed direction. This is the condition which is more or less perfectly realized in the instruments invented by S'gravesande, Gambey, Silbermann, Foucault, and others.

If the beam reflected by the mirror thus guided is received by a telescope along its principal axis, the focal image of the heavenly body will remain fixed in the center of the field of view in spite of the angular displacement of the celestial sphere. But this condition of fixity, geometrically realized for the object under examination, is no longer fulfilled for the neighboring regions: it is easily shown that the field of view turns about its center in such a way as to effect a complete revolution n twenty-four hours. The velocity of rotation is not uniform, and thus the angular displacement of the field varies with the time in accordance with a law which it is important to determine.

Let us represent the celestial sphere by a sphere of unit radius, and each line of sight on the sky by the projection on this sphere of a straight line passing through the center parallel to this direction.

Let *NESW* be the circle of the true or apparent horizon (Fig. 1); *P* the celestial pole; *Z* the zenith; *PZS* the meridian of the place; *PD* the hour circle of the star *D*, and *D'* the point of the true or apparent horizon toward which the reflected beam is constantly directed.

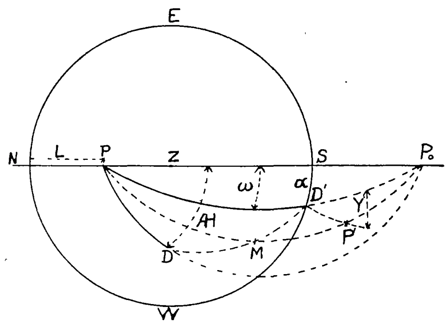

FIG. 1.

The position of the star *D* is defined at any instant by its polar distance $\delta = PD$, and its hour angle $AH = SPD$ counted positively in the direction of the diurnal motion, from the east *E* toward the west *W*. Similarly the point *D'* is determined by its polar distance $\rho = PD'$ and by the angle $\omega = SPD'$, which the plane *PD'* makes with the meridian. We will call this plane *SPD*, which is the hour circle passing through the point *D'* extended, the *reference plane*.[1]

If in place of ρ and ω there were given the azimuth $\alpha = SD'$ and the arc *PS*, the supplement of the latitude *L*, ρ and ω would be calculated by the aid of the two following expressions furnished by the right triangle *PSD'*:

$$\cos \rho = \cos \alpha \cos L, \quad \tan \omega = \frac{\tan \alpha}{\sin L}.$$

In order that the beam coming from the star *D* shall be constantly reflected to *D'*, it is necessary and sufficient, according to the laws of reflection, that the projection *M* of the normal to the mirror shall be maintained by the mechanism at the middle of the arc of the great circle *DD'*. Knowing at any instant the

[1] The use of stereographic projection on the circle of the horizon permits all of these arcs of circles to be rigorously traced out: it is well to adopt it in order to verify graphically the size and direction of the calculated angles. But this manner of projection has the inconvenience of so greatly distorting the sides of the spherical triangles which rise from the circle of the horizon that the use of these projections is more troublesome than useful for the clearness of the demonstrations. For this reason schematic figures are employed here instead of any regular system of projection.

projection M of this normal, we can draw the projection of the direction in which a ray emitted from any point whatever of the celestial sphere is reflected by the mirror: it is only necessary to join this point to the point M by an arc of a great circle and to prolong this arc through an equal distance. Thus the image P' of the pole P is on the arc PM prolonged to the point P', so that $MP' = MP$. The reflected spherical image of the various points of the celestial sphere is thus at any instant symmetrical to their direct position with respect to the point M.

It follows from this that the orientation of the field of view is wholly determined by the knowledge of the reflected image of any given point except that occupied by the star at the center. The pole P, because of its fixed position on the celestial sphere, is particularly adapted for this purpose, and its image P' constitutes a most simple and convenient standard of comparison.

We will now calculate for each instrument the distance and the orientation of the image P' of the pole, *i. e.*, the length of the arc $D'P'$ and the angle Y made by this arc with the great circle $PD'P_0$, the projection of the *reference plane*.

Siderostat.—This name is used to designate the apparatus specially constructed for the purpose of sending a reflected beam toward the southern horizon.

The advantage of this arrangement, which was devised by Léon Foucault, is to reduce as far as possible the angle of incidence $DM = D'M$ of the rays coming from stars which at their upper transit culminate near the zenith or the equator: the aberrations of the reflected image caused by imperfections of the mirror are thus materially reduced. Fig. 1 represents the course of the beam coming from the star D and sent by the siderostat in a horizontal direction; it makes with the southern meridian an angle α, which is counted positively toward the west; α is generally a small fraction of a right angle.

1. *Distance $D'P'$ from the image P' of the pole to D', the center of the field.* The arc $D'P'$ is the side of the triangle $MD'P'$ symmetrical with the triangle MDP, since $MD' = MD$

and $MP' = MP$. These two triangles are equal, as they have an equal angle at M included between two equal sides. The two sides $D'P'$ and DP opposite the equal angle are thus equal: $D'P' = DP = \delta$. Thus *the distance* $D'P'$ *from the image of the pole to the image of the star* (*center of the field*) *is equal to the polar distance of the star observed.* From this it follows that *the image of the pole describes about the center of the field a circle with a radius equal to the polar distance of the star observed.*

2. *Orientation of the arc* $D'P'$.—Let Y be the angle which the arc $D'P'$ makes with $D'P_0$, the prolongation of the projection of the reference plane DP. $Y = P_0D'P' = \pi - PD'P' = \pi - (PD'\ D + DD'\ P') = \pi - (PD'\ D + PDD')$, for $DD'\ P' = PDD'$ on account of the equality of the triangles MDP and $MD'P'$. The desired angle Y is thus the supplement of the angles at the base of the triangle PDD', the apex of which is at P. From Neper's formula

$$\tan \frac{1}{2}(B + C) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)} \cot \frac{A}{2},$$

we have, by substituting $A = DPD' = AH - \omega$, $b = \rho$, $c = \delta$,

$$\tan \frac{1}{2} Y = \frac{\cos \frac{1}{2}(\rho - \delta)}{\cos \frac{1}{2}(\rho + \delta)} \tan \frac{1}{2}(AH - \omega),$$

an expression which gives the orientation of the arc $D'P'$ and hence the law of rotation of the field of view, as AH varies proportionally to the time.

If we take as the origin of time the moment when the observed star is in the reference plane, $t = 0$ for $AH - \omega = 0$, and for the unit of time the sidereal or solar day (depending upon the object observed), we have $AH\omega = 2\pi t$ and the expression for Y takes the symmetrical form

$$\tan \frac{1}{2} Y = K \tan \frac{1}{2} 2\pi t,$$

where

$$K = \frac{\cos\frac{1}{2}(\rho - \delta)}{\cos\frac{1}{2}(\rho + \delta)} \text{ and } AH - \omega = 2\pi t.$$

It immediately follows :

a. That the rotation of the field has the same period as the diurnal motion.

b. It is continuous and always in the same direction, direct or inverse, according to the sign of K.

c. The reference plane is a plane of symmetry, for the angle Y takes equal values with contrary signs for equidistant epochs on opposite sides of the origin of time.

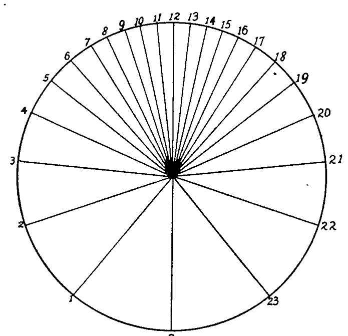

FIG. 2.

This law of rotation might be represented geometrically by a curve plotted with the time as abscissa and the angle Y as ordinate. But a more direct configuration of the rotation of the field may be obtained by considering the arc $D'P'$ as the moving radius vector of the circle described by P', the image of the pole, and by tracing the successive divisions of this radius vector at equidistant epochs, aliquot parts of a day. Fig. 2 is a representation of this character on the plane tangent to the sphere at D': the twenty-four successive positions of $D'P'$ are projected as straight lines ; they correspond to a subdivision of the day into twenty-four hours. The origin of time $t = 0$ corresponds to $D'P_0$, the projection of the reference plane and axis of symmetry.

3. *Expression for the angular velocity.*—The angular velocity of rotation at the epoch t is obtained by taking the derivative of the expression for Y with respect to t; after making the necessary reductions we obtain the formula

$$\frac{dY}{dt} = 2\pi \frac{K}{\cos^2 \pi t + K^2 \sin^2 \pi t}.$$

The denominator is necessarily positive, the velocity always having the sign of K; it varies periodically between the value $2\pi K$, corresponding to the epochs $t=0, 1, 2, \ldots$ and the value $\frac{2\pi}{K}$, corresponding to the intermediate epochs $t=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, ... passing through the value 2π, the angular velocity of the diurnal motion, at the epochs given by the condition

$$\cos^2 \pi t + K^2 \sin^2 \pi t = K$$

or

$$\tan \pi t = \frac{\pm 1}{\sqrt{1+K}}.$$

As the positions of the star which are most favorable for observation (upper transit) are near the reference plane $t = 0$, the velocity of rotation may be considered as constant and equal to $2\pi K$, for this velocity varies but little in the neigborhood of $t=0$, since it corresponds to a maximum or a minimum. The velocity $\frac{2\pi}{K}$ is never realized with the siderostat, with which lower transits cannot be observed.

The unit of angular velocity is evidently 2π or one circumference per day; if another unit is preferred, for example, to express the velocity in minutes of arc per minute of time or more generally in n^{ths} of a circumference per m^{ths} of a day, it is only necessary to substitute $\frac{n}{m}$ for 2π. This change of units amounts to the same thing as placing

$$\frac{Y}{2\pi} = \frac{y'}{n}, \qquad \frac{t'}{1} = \frac{\pi}{m},$$

whence

$$\frac{dy'}{d\pi} = \frac{1}{2\pi} \frac{n}{m} \frac{dY}{dt}.$$

As there are $n = 360 \times 60$ minutes of arc in the circumference and $m = 24 \times 60$ minutes of time, the velocity $2\pi K$, or $\frac{n}{m} K$, is here equal to $15 K$; we thus find $15'$ of angle per minute of time for the angular velocity of the diurnal motion $K=1$.

4. *Direction of rotation of the field of view.*— Let us suppose that the observer is receiving the luminous beam; he thus looks

toward the center of the sphere along the radius which terminates at D', whence it results that the direction of the motion of rotation will be that which an observer placed in the direction D' outside of the sphere will attribute to the motion of the arc $D'P'$. From the expression for Y it is evident that Y and $RA = \omega$ will be of the same sign if the coefficient K is positive. The direction of the diurnal motion, *i. e.*, the direction of the positive variation of RA, is known; it is seen in the figure that when the right ascension of the star D increases the arc PD, seen from outside the sphere, turns in the direction of the hands of a watch; thus for a positive value of K, Y varies in the same manner. The condition under which K is positive is evidently

$$\cos \frac{1}{2}(\rho + \delta) > 0 , \quad \frac{1}{2}(\rho + \delta) < \frac{\pi}{2} , \quad \delta < \pi - \rho .$$

Whence we conclude

When the polar distance of the observed star is less than the supplement of the polar distance of the reflected direction, the apparent direction of rotation of the field of view of the siderostat is that of the hands of a watch.

It is in the contrary direction if the polar distance of the star is less than this supplement. Observation with an astronomical telescope does not change the direction of rotation: the *reversal* of the image is confined to turning through 180° the direction of the origin $D'P_0$.

5. *Critical polar distance: fixed field of view.*—The point of transition between these two cases corresponds to the condition $K = 0$, that is $\cos \frac{1}{2} (\rho + \delta) = 0$; the value of Y remains constantly zero, whatever be the right ascension of the star. Hence

The field of view of the siderostat remains rigorously fixed when the polar distance of the observed star is equal to the supplement of the polar distance of the direction of reflection.

This case of absolute immobility of the field has a corresponding geometrical peculiarity which renders the result evident. It is in fact easily shown that if $\rho + \delta = \pi$, the arc $PM = \frac{\pi}{2}$; the normal to the mirror becomes normal to the line joining the poles, and thus the mirror is parallel to the Earth's axis. Furthermore the arc PM bisects the angle DPD', and consequently the mirror turns through an angle equal to half the change in the hour angle. These are the two characteristic conditions of the *Coelostat* of M. Lippmann, a very simple instrument which gives an absolutely fixed image of the sky. It consists of a mirror turning about an axis parallel to its own plane and to the Earth's axis with an angular velocity equal to half that of the diurnal motion and in the same direction.

The siderostat may thus replace the coelostat for a region of the sky surrounding a star of polar distance δ. It is only necessary to send the reflected beam in a direction such as to satisfy the condition $\rho + \delta = \pi$, *i. e.*, along one of the generators of the cone of revolution which makes with the Earth's axis the angle $\pi - \delta$, the supplement of the polar distance. It is well to be acquainted with this property of the siderostat, for in certain cases it can be used without serious inconvenience.

6. *Siderostat oriented in the meridian.*— This is the most common arrangement of the siderostat: the horizontally reflected beam is directed exactly toward the south. It follows from this that $\omega = 0$, $\rho = \pi - L$, L being the latitude. The angle Y is the angle which the arc $D'P'$ makes with the meridian, which now becomes the plane of reference and of symmetry. The expression for Y takes the form

$$\tan \frac{1}{2} Y = K \tan \frac{1}{2} AH,$$

where

$$K = \frac{\sin \frac{1}{2}(L - \delta)}{\sin \frac{1}{2}(L + \delta)}.$$

The above propositions then become very simple.

With the siderostat oriented in the meridian, the field of view is fixed when the polar distance of the observed star is equal to the latitude of the place of observation. The rotation of the field is in the direction of the hands of a watch if this polar distance is less than the latitude; in the inverse direction if it is greater.

The coefficient K, which defines the practically constant value of the velocity of rotation of the field as a function of the diurnal rotation, is always less than unity.

The following numerical values are for the latitude of Paris $L = 48° \ 50'$, δ increasing by successive steps of 10° (upper transit).

δ	K	δ	K
0° (pole)	1.000	70°	−0.213
10	0.677	80	−0.298
20	0.440	90	−0.376
30	0.258	100	−0.448
40	0.105	110	−0.518
48 50 L	0.000	120	−0.585
50	−0.013	130	−0.651
60	−0.120		

Heliostat.—This instrument sends the reflected beam in the direction of the northern horizon, rarely beyond N. E. or N. W. Fig. 3 represents the beam coming from the star D and sent in the horizontal direction D'', making with the north meridian an angle $NPD'' = \alpha'$, counted positively toward the east. The pole is at P, SPD is the hour angle and δ the polar distance of the observed star D. We shall designate by $\omega' = NPD''$ and $\rho' = PD''$ the hour angle and the polar distance of the direction of reflection D'', which may be deduced as before from the azimuth α' and the latitude L by means of the right triangle NPD''. The normal M to the mirror is at the middle of the arc DD'', and the image P'' of the pole is on the arc PM produced so that $MP'' = MP$.

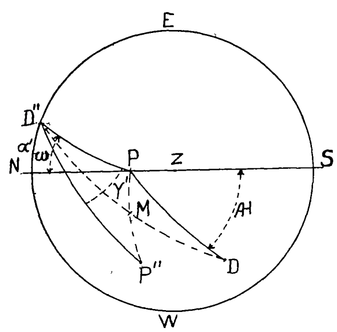

FIG. 3.

The angle of rotation of the field will be determined by the angle made by the arc $D''P''$ with the projection of the reference plane PD''.

Let $Y' = PD''P''$ be this angle; it might easily be deduced from the expression for Y for the siderostat, which is similarly defined. But it is simpler to calculate it directly.

The triangles PDM and $P''D''M$ are equal, as they have an equal angle at M included between two equal sides $DM = D''M$, $PM = P''M$. Thus $D''P'' = PD = \delta$.

Thus, as in the case of the siderostat, *the image of the pole reflected by the heliostat describes about the center of the field a circle with a radius equal to the polar distance of the observed star.*

Furthermore,

$$Y' = PD''P = PD''D + DD''P'' = PD''D + D''DP',$$

in consequence of the equality of the two triangles PMD and $P''MD''$. The angle Y' is thus the sum of the two angles at the base of the triangle PDD'', whose apex is at P. From Neper's formula referred to above we obtain, after substituting

$$b = \rho'', \quad c = \delta, \quad \text{and } A = \pi - AH + \omega',$$

$$\tan \frac{1}{2} Y' = \frac{\cos \frac{1}{2} (\rho' - \delta)}{\cos \frac{1}{2} (\rho' + \delta')} \tan \frac{1}{2} (AH - \omega').$$

Y is counted positively in the direction of the hands of a watch. This expression may also be put in the form

$$\tan \frac{1}{2} Y' = K' \tan \frac{1}{2} 2\pi t,$$

where

$$K' = \frac{\cos \frac{1}{2} (\rho' - \delta)}{\cos \frac{1}{2} (\rho' + \delta)} \quad \text{and} \quad AH - \omega' = 2\pi t.$$

Thus we again obtain the three conclusions (a), (b), (c), demonstrated above for the siderostat. It is unnecessary to repeat the discussion, which would be quite similar; but we wish to call special attention to the practical points of difference between the two instruments. With the heliostat the coefficient K' is always greater than unity and retains the positive sign under the conditions in which the heliostat is ordinarily employed, *i. e.*, in the observation of the upper transit of heavenly bodies near the zenith or the equator, reflected in a direction which does not greatly differ from that of the northern horizon.

By developing the value of the cosine we may write K' in the form

$$K' = \frac{1 + \tan\frac{1}{2}\rho' \tan\frac{1}{2}\delta}{1 - \tan\frac{1}{2}\rho' \tan\frac{1}{2}\delta}.$$

The + sign of the coefficient K' here corresponds, as may be seen in the figure, to a variation in Y' contrary in direction to that of the hour angle AH. Hence we conclude that

The field of view of the heliostat, under the actual conditions of observation, turns with an angular velocity which is always greater than that of the diurnal motion; the direction of rotation is that of the hands of a watch.

This conclusion puts in evidence a new cause for the inferiority of the heliostat as compared with the siderostat. To the inconvenience arising from the reflection at great angles of incidence from the mirror of the heliostat must be added that of a great velocity of rotation of the field of view. These two conditions are unfavorable for observations which require in the images both great perfection and complete stability. On account of this fact the siderostat is to be preferred for astronomy of precision.

But this rapidity of rotation of the field is not always an inconvenience; for certain astrophysical observations it is on the contrary advantageous, in that it dispenses with the necessity of employing complex and delicate optical arrangements; here is an example.

Suppose we project, with the aid of a suitable objective, the solar image reflected by a heliostat on to the slit of a spectroscope of high dispersion for the purpose of studying the displacement of lines due to the motions of the solar surface. The most favorable condition occurs when the solar equator is normal to the slit; if the image is oscillated in such a way as to make the opposite limbs of the disk successively tangential to the slit, one may obtain twice the maximum displacement due to the difference of the radial velocities at the equator (method of oscillating lines).

Except under unusual circumstances the image of the solar disk will not be found in this favorable azimuth and will have little chance of attaining it if a siderostat is employed, since with this apparatus the velocity of rotation of the field of view is zero or very small.

In order to bring the equator to the required azimuth it is necessary to make use of an auxiliary apparatus consisting, for example, of an isosceles total reflection prism, movable about an axis parallel to its base; rotation of this prism changes the azimuth of the Sun's disk by twice the angle, which permits the equator to be placed perpendicular to the slit in the two successive positions 180° apart, which give the maximum double displacements in the inverse order. But the prism must be very perfect both as to material and flatness of the surfaces. Moreover, the rotating mounting which carries it is rather difficult to construct and adjust.

With the heliostat, the natural rotation of the field of view renders this auxiliary apparatus unnecessary; it is sufficient to await the effect of this rotation and to allow the solar equator to place itself perpendicular to the slit. At certain times of the year and for certain orientations of the slit and of the beam reflected by the heliostat, this condition of perpendicularity occurs twice in the same day within an interval of a few hours, the image of the solar equator turning through 180°.

This result, which I discovered experimentally and observed on several occasions, greatly surprised me at first; I had supposed that about twelve hours would be required for the reflected image of the solar disk to turn 180° about its center. The search for an explanation of this phenomenon led to the preparation of this paper. The complete discussion would require rather extended developments. I shall confine myself here to indicating the principle of the demonstrations.

The explanation is based upon the relative velocity of the field of the heliostat when the observed star is near the equator

($\delta = 90°$). The coefficient K', which measures it as a function of the diurnal rotation, varies between 2 and 5 for positions of the Sun between the two solstices.

The accompanying table gives the values of K' in the usual case where the heliostat is oriented in the meridian, the reflected beam being directed horizontally toward the north; we substitute in the formula $\omega' = 0$, $\rho' = L = 48° \ 50'$, δ increasing by steps of 10° (upper transits).

δ	K'	δ	K'
0° (pole) - -	1.000	70° - - - - -	1.932
10 - - - -	1.083	80 - - - -	2.231
20 - - - - -	1.174	90 (equinox) - -	2.663
30 - - - -	1.277	100 - - - -	3.358
40 - - - - -	1.396	113 37 (winter solstice)	5.489
50 - - - -	1.537	110 - - - -	4.687
60 - - - - -	1.710	120 - - - - -	8.359
66 38 (summer solstice)	1.849		

The value of K' approaches infinity, which it attains when the star is at the southern horizon; this is in fact a critical polar distance $\rho' + \delta = \pi$, which corresponds to grazing incidence at the mirror.

If we know the value of K' we can calculate the time which elapses between the epochs t_1 and t_2, between which the image of the field has turned through 180°. Let Y_1' be the value of the angle Y' at the epoch t_1, when the solar equator, for example, is normal to the slit of the spectroscope, and $Y_2' = Y_1' + \pi$, the value of Y_1' increased by 180° at the epoch t_2. We shall then have the two conditions

$$\tan \frac{1}{2} Y_1' = - K' \tan \pi t_1 \quad \cot \frac{1}{2} Y_2' = K' \tan \pi t_2 .$$

Multiplying member by member we finally obtain

$$\tan \pi t_1 \tan \pi t_2 = - \frac{1}{K'^2} .$$

The minus sign shows that the two epochs t_1 and t_2 (supposed to be as near together as possible) are of contrary sign, which signifies that the two corresponding positions of the star are situated on opposite sides of the reference plane (in this case the plane of the meridian); it is necessary to except the limiting cases for which $t=0$ and $t=\frac{1}{2}$. Let $\theta=t_2-t_1$ be the difference between the two epochs; if t_1 is given t_2 may be calculated. It is especially interesting to find the two epochs for which this difference is a minimum.

Let us therefore make $d\theta=0$, that is, $dt_2-dt_1=0$, and let us differentiate the expression which connects t_1 and t_2; we obtain, after completing the operation,

$$\sin\pi(t_2+t_1)\cos\pi(t_2-t_1)=0\,.$$

This is the solution $t_2+t_1=0$ which gives the desired minimum; the other, $t_2-t_1=\frac{1}{2}$, gives the 12-hour maximum, which is of no interest.

The two desired epochs symmetrical with respect to $t=0$ are of equal length and contrary sign; substituting, in order to obtain their absolute values, we have

$$\tan\pi t=\frac{1}{K'}\,.$$

If we give K' increasing values starting from $K'=1$ (uniform rotation), which gives $t=\frac{1}{4}$ of a day, or 6 hours, and $t_2-t_1=$ 12 hours, we find that the interval $\theta=t_2-t_1$ grows smaller and smaller. Making the calculation to determine this difference θ for the three most interesting epochs relating to the Sun we obtain

	θ
Summer solstice - - - - - - - -	$7^h\ 54^m$
Equinox - - - - - - - -	5 30
Winter solstice - - - - - - - -	2 45

results which demonstrate the possibility of seeing the solar equator turn through 180° in much less than 12 hours.

It is, however, unnecessary that the rotation should be exactly 180° in order to show successively the two inverse effects of the oscillating lines, for the absolute velocity of the solar parallels diminishes only $\frac{1}{10}$ up to $\pm 25°$ heliocentric latitude, so that a displacement of $180°-50°=130°$ is sufficient to show the double phenomenon in the clearest manner.

It remains to determine the orientations of the reflected beam which are most favorable for observation; but this problem is rather complex and deserves to be treated separately.

What precedes is sufficient to show that even in those peculiarities of instruments which at first sight appear to be unfortunate imperfections, possibilities exist which may be turned to account in other classes of work. The complete study of the geometrical properties of instruments commonly reveals some peculiarity capable of rendering unexpected services.

PARIS,
January 1900.