Order of Operations, plus tidbits on handling equations

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Parentheses **E**xponents **M**ultiplication **D**ivision **A**ddition **S**ubtraction

Most math students are familiar with this order of operations and some way to remember it. This is good enough for basic math, but not for algebra. These notes should explain the nuances of order of operations that are important for mainstream algebra.

1. Symbols for multiplication

a) The " x " symbol. This should be avoided after basic math, because it looks like the variable "x".

b) The " · " symbol. The dot should be written high enough that it does not look like a decimal point.

c) Juxtaposition (putting beside). This is the usual way to write multiplication when variables are involved. Examples: $2x = 2 \cdot x$; $4abc = 4 \cdot a \cdot b \cdot c$

2. Symbols for division

a) The " ÷ " symbol. This is rarely used after basic math.

b) The " / " symbol. The slash means the same as the ÷ symbol, and is commonly used on calculators and computers.

c) The fraction bar. This is usually the preferred way to write division, because the fraction bar is a grouping symbol that is easier to read than parentheses. More on this below.

3. Addition and subtraction have the same priority.

Subtraction is adding a negative number, so it is equal to addition for purposes of order of operations. One way to give the same level of priority to addition and subtraction is to *work both operations as they occur from left to right*. For example:

Incorrect Method (add then subtract)	Correct Method (both, left to right)
$2 + 3 - 4 + 8 - 5$	$2 + 3 - 4 + 8 - 5$
$5 \quad - 4 + 8 - 5$	$5 \quad - 4 + 8 - 5$
$5 \quad - \quad 12 \quad - 5$	$1 \quad + 8 - 5$
$-7 \quad - 5$	$9 \quad - 5$
-12	4

Doing all the addition first causes the 4 and 8 to group together, and subtracts them both. Because 8 is subtracted from the final result instead of added, the total is 16 less than it should be. In symbols, the incorrect method does this:

$$2 + 3 - (4 + 8) - 5$$

Another method for handling subtraction is to rewrite it as addition of a negative. This eliminates the visual distinction between addition and subtraction operations. The above problem can be rewritten as:

$$2 + 3 + (-4) + 8 + (-5)$$

3. Multiplication and division have the same priority.

Division is multiplying by the reciprocal, so it is equal to multiplication for purposes of order of operations. To solve a simple problem that has multiplication and division, *work both operations as they occur from left to right*. For example:

Incorrect Method (multiply then divide)	Correct Method (both, left to right)
$20 \div 4 \cdot 10 \div 2$	$20 \div 4 \cdot 10 \div 2$
$20 \div 40 \div 2$	$5 \cdot 10 \div 2$
$0.5 \div 2$	$50 \div 2$
0.25	25

Doing all the multiplication first causes the 4 and 10 to group together, and divides by their product. Because the final result is divided by 10 instead of multiplied by 10, it is too small by a factor of 100. In symbols, the incorrect method does this:

$$20 \div (4 \cdot 10) \div 2 \qquad \text{or} \qquad \frac{\;\frac{20}{4 \cdot 10}\;}{2}$$

4. Exponents and radicals have the same priority.

This is because a radical, or root, denotes a power and can be written as an exponent. This should make sense because raising to a power can be reversed by taking a root. For example:

$$6^2 = 36 \qquad 4^3 = 64 \qquad 3^5 = 243$$

$$\sqrt{36} = 36^{(1/2)} = 6 \qquad \sqrt[3]{64} = 64^{(1/3)} = 4 \qquad \sqrt[5]{243} = 243^{(1/5)} = 3$$

5. Grouping symbols

a. Parentheses “( )”, brackets “[ ]”, and braces “{ }”

For purposes of order of operations, these symbols mean the same thing. Whatever is enclosed in parentheses must be simplified first, or if not, *it must interact as a quantity with anything outside*. This means any operation on the quantity in parentheses must be applied to all of its parts. The most common and important example is called the distributive property:

$$a(b + c) = ab + ac \qquad 2(7 - d) = 14 - 2d$$

When polynomials are multiplied, they interact as quantities and must be enclosed in parentheses. Binomials can be multiplied using the FOIL method, which pairs each term of one binomial with each term of the other. FOIL = **F**irst terms, **O**utside terms, **I**nside terms, **L**ast terms. For example:

$$(2x + 3)(x - 5) = \underset{\substack{\uparrow \\ \text{First}}}{2x \cdot x} + \underset{\substack{\uparrow \\ \text{Outside}}}{2x \cdot -5} + \underset{\substack{\uparrow \\ \text{Inside}}}{3 \cdot x} + \underset{\substack{\uparrow \\ \text{Last}}}{3 \cdot -5} = 2x^2 + -10x + 3x + -15 = 2x^2 - 7x - 15$$

It's good to realize that the FOIL method, and the idea of multiplying polynomials by matching all terms, result from the distributive property. This is more clear when the above example is done this way:

$$(x - 5)(2x + 3) \;=\; \mathbf{(x - 5)2x \;+\; (x - 5)3} \;=\; x \cdot 2x \;+\; -5 \cdot 2x \;+\; x \cdot 3 \;+\; -5 \cdot 3$$
$$= \;2x^2 \;+\; -10x \;+\; 3x \;+\; -15 \;=\; 2x^2 - 7x - 15$$

Here the first step (in bold) shows how the distributive property -- $\mathbf{a(b + c) \;=\; ab + ac}$ -- is used. In this example,

$a = (x - 5)$ $\qquad$ $b = 2x$ $\qquad$ $c = 3$

The distributive property is also used twice in the second step of the above solution. This is more clear when the order is changed to match $a(b + c)$:

$$2x(x - 5) \;+\; 3(x - 5) \;=\; (\,2x \cdot x \;+\; 2x \cdot -5\,) + (\,3 \cdot x \;+\; 3 \cdot -5\,)$$

When $2x$ and 3 are each distributed across $(x - 5)$, the same result is produced as in the FOIL method. This shows that FOIL is a shortcut to work through two levels of applying the distributive property. Math shortcuts are good, but it's important to understand why they work. This one works by the distributive property.

Using the distributive property in reverse is called *factoring*. Here are some examples:

$$d^3 + d \;=\; d(d^2 + 1) \qquad 30x^3 + 40x^2 \;=\; 10x^2(3x + 4) \qquad 0.5y^2 - 0.25y \;=\; 0.25y(2y - 1)$$

$$3x^2 + 5x - 12 = (3x - 4)(x + 3) \qquad x^3 - 6x^2 + 8x \;=\; x(x + 2)(x + 4)$$

Factoring methods are beyond the scope of these notes, but I will mention one case where factoring is obviously the distributive property in reverse: *factoring by grouping*. It's helpful in algebra to understand this concept, which can be tricky.

Consider this cubic polynomial: $\qquad 2x^3 + 7x^2 - 6x - 21$

There is no common factor of all four terms. In other words, there is no integer, and no positive integer power of x, that will neatly "undistribute" from the polynomial. Anything *could* be taken out as a factor, but this would only make the expression more complicated. For example:

$$2x\left(x^2 + \frac{7}{2}x - 3 - \frac{21}{2x}\right) \qquad \text{or} \qquad 4a\left(\frac{1}{2a}x^3 + \frac{7}{4a}x^2 - \frac{3}{2a}x - \frac{21}{4a}\right)$$

Although no factor of all four terms will help, it *is* helpful to factor the terms two by two, like this:

$$2x^3 + 7x^2 - 6x - 21 \;=\; x^2\,(\,2x + 7\,) \;+\; -3\,(\,2x + 7\,)$$

Because the same binomial, $2x + 7$, is left when the two factors are "undistributed", you can now use the distributive property in reverse a second time, like this:

$$x^2\,(\,2x + 7\,) \;+\; -3\,(\,2x + 7\,) \;=\; (x^2 - 3)(2x + 7)$$

This whole process is simply the FOIL method in reverse. It's more difficult in reverse because you have to recognize when factoring by grouping is possible, and when it's not. This is done by looking for a common ratio between the first two terms and last two terms (in descending order). For example:

$$2x^3 + 7x^2 - 6x - 21 \qquad \text{ratios: } \frac{2x^3}{7x^2} = \frac{2x}{7}\ , \ \frac{-6x}{-21} = \frac{2x}{7}$$

The common ratio of $2x : 7$ shows that factoring by grouping will work.

Parentheses are only obligatory in a few situations. Otherwise, they are unnecessary but can sometimes make an expression more clear.

The associative properties of addition and multiplication say that parentheses are not needed in these cases:

$$(a + b) + c = a + (b + c) = a + b + c \qquad (ab)c = a(bc) = abc$$

These are the usual situations in which parentheses are necessary in handwritten math:

1. Multiplication by a sum or difference. For example: $a(b + 3)$ $\quad (2 - d) \cdot x$
2. Division using the ÷ or / sign, when there is an operation in the numerator or denominator. This is discussed in section c on the next page.
3. Raising to a power anything other than a single variable, function, or positive number.

When parentheses are not present, an exponent applies only to the item directly to its left, not including the negative sign if present. Thus most expressions must be placed in parentheses for a single exponent to be applied to them. For example:

$$ab^3 = a \cdot b^3 = a \cdot b \cdot b \cdot b \qquad (ab)^3 = a^3b^3 = a \cdot a \cdot a \cdot b \cdot b \cdot b$$

$$-c^2 = -1 \cdot c^2 = -1 \cdot c \cdot c \qquad (-c)^2 = -c \cdot -c$$

$$2x + y^2 = 2x + y \cdot y \qquad (2x + y)^2 = (2x + y)(2x + y) = 4x^2 + 4xy + y^2$$

Parentheses, brackets, and braces are purely grouping symbols; that is, they do not denote any operation in addition to grouping. The following symbols indicate grouping *in addition to* or *as a consequence of* their primary meaning.

b. Equality and inequality signs

These signs make statements about the entire quantities on their two sides. This implies the grouping of each side. For example:

$$\sqrt{y+4} = y - 3 \qquad \text{and} \qquad \frac{b}{4} - 5 \le 3 - \frac{1}{b}$$

may be written as $(\sqrt{y+4}) = (y-3)$ and $(\frac{b}{4} - 5) \le (3 - \frac{1}{b})$

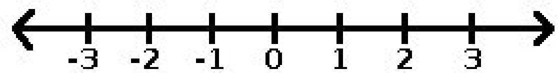
This fact is very important when operations are done to both sides of an equation or inequality, as commonly occurs in manipulating and solving equations. These operations must be applied to *the entire quantity* on each side of the equation, like this:

$$(\sqrt{y+4}\,)^2 = (y-3)^2 \qquad\qquad b\left(\frac{b}{4} - 5\right) \le b\left(3 - \frac{1}{b}\right)$$

FOIL method $\quad y+4 = y^2 - 6y + 9 \qquad\qquad \left(\frac{b^2}{4} - 5b\right) \le (3b - 1)\quad$ distributive property

Multiplication or division by a negative number or quantity changes the direction of the inequality sign. This makes sense when you consider a simple example:

$2 < 3$	$-1 \cdot 2 > -1 \cdot 3$	$-2 > -3$
2 is left of 3	change sign	-2 is right of -3



c. Fraction bars

Fraction bars break an expression into a clearly defined numerator and denominator, each of which must be grouped together for the division. This often eliminates parentheses that are needed to write the same expression with a ÷ or / sign. For example:

$$\frac{x^2+3}{x} = (x^2 + 3) / x \qquad\qquad \frac{a^3-4}{1-a^2}\cdot\frac{1+a}{a+2} = ((a^3 - 4) / (1 - a^2)) \cdot ((1 + a) / (a + 2))$$

The rule for using the ÷ or / sign is that the dividend and divisor (numerator and denominator) are each limited to the number, variable, function, or parenthetical quantity directly adjoining the ÷ or / operator, with its exponent or root. This is most important for denominators. If this rule is confusing, just make it a habit to always put in parentheses your numerator, denominator, and anything else you think may need them.

These sample expressions are written identically with and without a fraction bar.

$$1 / 2x = \frac{1}{2}\cdot x \qquad 1 / (2x) = \frac{1}{2x} \qquad 5 / c^2 = \frac{5}{c^2} \qquad f(x) / x = \frac{f(x)}{x}$$

$$x + 1 / x - 3 = x + \frac{1}{x} - 3 \qquad (x + 1) / (x - 3) = \frac{x+1}{x-3} \qquad \sin(x-3)^2 / \sqrt{x} = \frac{\sin^2(x-3)}{\sqrt{x}}$$

This piece of trivia surprised me when I switched calculators: The TI-82 applies a different grouping rule for division. On the TI-82,

$$1 / 2x = \frac{1}{2x}$$

This doesn't affect many students today, but TI-82 users should read the manual and get all the details about it. When in doubt, use parentheses.

d. Absolute value signs

Absolute value signs, or bars, are a grouping symbol because their effect applies to the entire quantity they enclose. For example, in $|x + y|$, it is not x or y that is made positive, but their sum.

To solve an equation with one or more variables inside absolute value bars, replace the bars with parentheses and add a plus or minus sign (±) outside. This represents the possibilities for the quantity inside to be positive or negative. Because plus and minus cannot be done simultaneously, the equation must be rewritten as two separate equations, one with the plus sign, one with the minus. Both signs are then distributed across the expression inside the parentheses. For example:

$$| x + 5 | = 2$$
$$\pm (x + 5) = 2$$
$$+(x + 5) = 2 \quad \text{and} \quad -(x + 5) = 2$$
$$x + 5 = 2 \quad \text{and} \quad -x - 5 = 2 \qquad \text{distribute the negative sign like -1}$$
$$x = -3 \quad \text{and} \quad x = -7$$

To find absolute value on the TI-83 and similar calculators, press MATH, scroll one step right to the NUM menu, and it's option 1. This takes the absolute value of whatever is enclosed in parentheses to the right of the "abs". For example,

$$| x^2 - 7 | = \text{abs}(x^2 - 7) \qquad\qquad \frac{|4x-5|}{x} = (\text{abs}(4x - 5)) / x$$

e. Radicals

Everything under a radical sign, called the radicand, must have the radical applied to it before or while it interacts with other parts of the expression. This means that an expression under a radical sign can be considered to be in parentheses.

$$\sqrt[3]{x - 7} = \sqrt[3]{(x - 7)} \qquad\qquad \sqrt{r^2 + 10} = \sqrt{(r^2 + 10)}$$

To rewrite a radical expression using exponents, enclose the radicand in parentheses and apply the exponent to the whole quantity. For example:

$$\sqrt{m - 5} = (m-5)^{(1/2)} \qquad\qquad \sqrt[3]{(n^2 - 1)^2} = ((n^2 - 1)^2)^{(1/3)} = (n^2 - 1)^{(2/3)}$$

f. Expressions as exponents

An expression written as an exponent is implicitly grouped together. It must interact as a quantity with other parts of the expression, according to the laws of exponents and logarithms. Examples:

$$5^{9-x} = 125 \quad \rightarrow \quad \log_5 125 = 9 - x \quad \rightarrow \quad 3 = 9 - x \quad \rightarrow \quad x = 6$$

$$\frac{2^{2x+3}}{2^{x-1}} = 2^{(2x+3) - (x-1)} = 2^{2x + 3 - x + 1} = 2^{x+4}$$

Exponents can be written on the TI-83 calculator using the carat (^) symbol, which is located between CLEAR and ÷ . On standard computer keyboards it is shift-6. For example:

$$2^4 = 2 \wedge 4 \qquad\qquad 4^{0.1} = 4 \wedge 0.1$$

Fractions and expressions in the exponent need to be enclosed in parentheses. For example:

$$5^{9-x} = 5 \wedge (9 - x) \qquad\qquad \frac{2^{2x+3}}{2^{x-1}} = (2 \wedge (2x + 3)) / (2 \wedge (x - 1))$$

6. Handling functions

Functions can be manipulated in an equation by placing them in parentheses and treating them like a variable. For example:

$$\begin{aligned} 9x + 3f(x) &= 24 \\ 9x + 3(\,f(x)\,) &= 24 \\ 3(\,f(x)\,) &= 24 - 9x \\ (\,f(x)\,) &= (24 - 9x) / 3 \\ f(x) &= 8 - 3x \end{aligned}$$

When you are comfortable with this, the extra parentheses around the function are not necessary.

